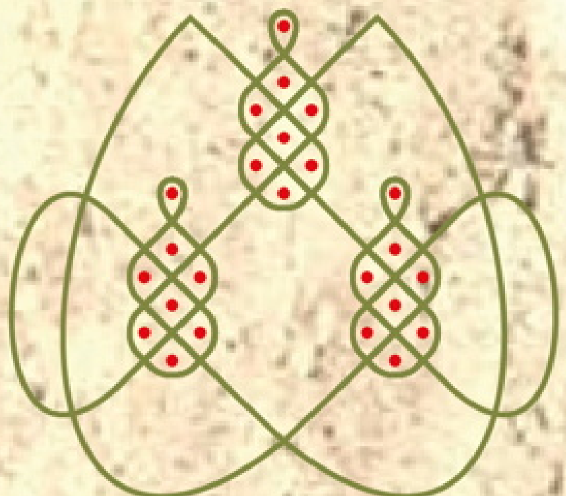
Paulus Gerdes

Sona Geometry from Angola

Volume 2

Educational and mathematical explorations of African designs on the sand



Translation and Foreword:
Darrah Chavey

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Volume 2:
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Translation: Darrah Chavey

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CHAPTER 3

EXPLORING THE MATHEMATICAL POTENTIAL OF *SONA*: AN EXAMPLE OF STIMULATING CULTURAL AWARENESS IN MATHEMATICS TEACHER EDUCATION [1]

3.1 Introduction: the need for a culture-oriented education

Development strategies that ignore or downplay the importance of cultural factors provoke indifference, alienation, and social discord, as highlighted in the *Report of the South Commission*, headed by the former President of Tanzania, Julius Nyerere (see Nyerere, 1991). Alternative development strategies should use the huge reserves of traditional knowledge, creativity, and sense of initiative in the countries of the Third World (Nyerere, 1991, p. 55). The Regional Consultation on Education for All (Dakar, November 27-30, 1989) concluded that Africa needs *culture-oriented education* (UNESCO, 1990, p. 6 & 15). The scientific appreciation of African cultural elements and experience is considered to be “one sure way of getting Africans to see science as a means of understanding their cultures and as a tool to serve and advance their cultures” (UNESCO, 1990, p. 23).

Educate or Perish: Africa’s Impasse and Prospects, a study led by the historian Joseph Ki-Zerbo, shows that currently African educational system – unadapted and elitist – promotes foreign consumption without generating a culture that is both compatible with the original civilization and truly promising. Africa needs a “new educational system, properly rooted in both the society and the

[1] Adapted version of a paper presented at the conference “The education of mathematics teachers in Southern Africa”, that took place during the 8th Symposium of the Southern African Mathematical Sciences Association, Maputo, 16-19 December 1991. Published as Chapter 9 in: Gerdes, Paulus (1995), *Ethnomathematics and Education in Africa*, Institute of International Education, University of Stockholm.

environment, and therefore apt to generate the *self-confidence from which imagination springs"* (Ki-Zerbo, 1990, p. 104).

3.2 Ethnomathematics research and teacher education

To avoid aversion — mathematical experience is seen as a rather bizarre and useless discipline of foreign origin and imported into Africa — the heritage, traditions and practices of mathematics in Africa (see Gerdes, 1992) must be "embedded", "integrated" or "incorporated" into the curriculum.

In order to gradually prepare an educational reform that ensures that real mathematics education "in tune with African traditions and socio-cultural environment" (UNESCO, 1990, p. 14) we started ethnomathematics research studies in Mozambique.

Ethnomathematics analyzes the connections between cultural development and mathematics education (see e.g. D'Ambrosio, 1990 and Gerdes, 1991), and more particularly studies:

* The mathematical traditions that have survived colonization, and math activities in the daily lives of people, seeking ways to integrate them into the school curriculum.
* The cultural elements that can serve as a starting point to develop mathematics at school and outside school.

Given that teachers play a fundamental role in (successful) curriculum reform, their training is a strategic time for debate and experimentation with 'cultural embedding of mathematics education'.

In our article *On culture, geometrical thinking and mathematics education* (1988) (reprinted in Gerdes, 1991), we gave some examples of "*cultural awareness raising*" of future teachers of mathematics: the construction of houses in Mozambique and the study of alternative axiomatic construction of Euclidean geometry; the weaving of funnels as a source of inspiration for discovering a general method for the construction of regular polygons; from interweaving of buttons to the "Pythagorean theorem"; from traditional traps for fishing to an alternative trigonometric function, tilings and the generation of regular and semi-regular polyhedra. This time, we would like to show how future teachers can carry out investigations by studying the cultural context and history of southern Africa, opening a new path in mathematical research. As an illustration, we will explore the mathematical potential of *sona*.

3.3 Examples of exploration of the mathematical potential of *sona* in teacher training

Sona offer a rich context for exploring mathematics. In general, it is important for future teachers of mathematics to understand what it means to do mathematics: that is to say, experiment, discover and formulate hypotheses, prove theorems. The study of *sona* and similar drawings provides — in Southern Africa — an appealing and culturally well-integrated context, where the sense of mathematics can be developed, as shown by the following examples.

Rules of composition

The (re-)discovery and demonstration of traditional Cokwe rules (as reconstructed) can be used to build up larger monolinear drawings enlarged from smaller monolinear drawings (cf. Vol. 1, Chapters 5 and 6). These rules are particularly adapted to research by future professors. Figure 3.1 shows a rule that was applied four times in the traditional representation of a leopard with her five small (see figure 3.2b).

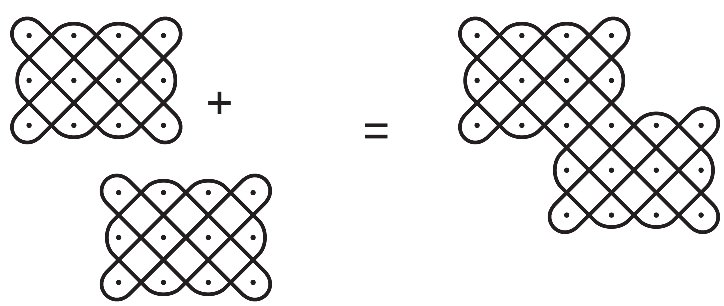
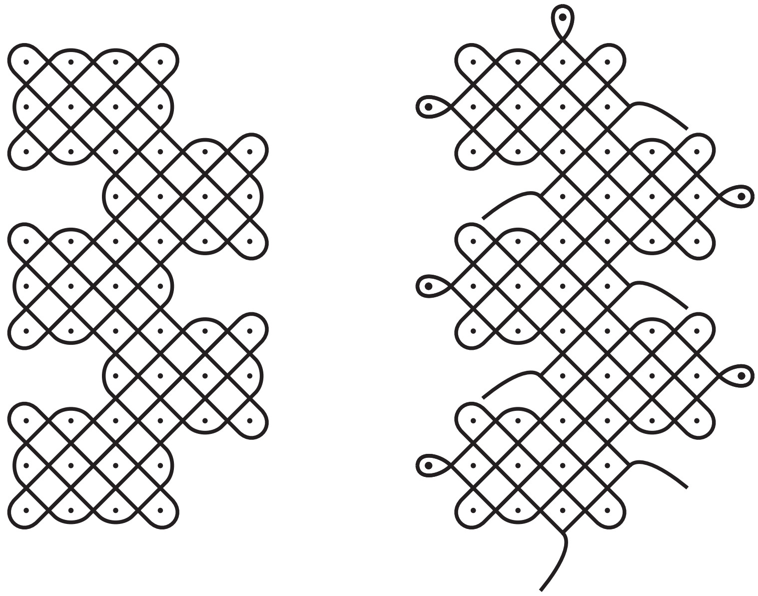


Figure 3.1



a b

Figure 3.2

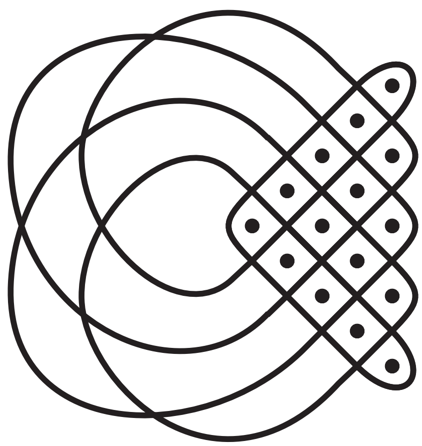


Figure 3.3

Systematic construction of monolinear drawings

* *First example*

Figure 3.3 shows the left half of a *lusona*. We can consider this to have been built from the "triangular pattern" of figure 3.4a, connecting the ends a, b, c, d, A, B, C, D according to the order DaAbBcCdD. Many questions, which encourage further consideration and a subsequent analysis, may arise, for example:

* Can we connect the ends of these monolinear drawings in a different order? (Figure 3.4b gives an example). If so, how many combinations are possible?
* How many ways can we construct monolinear drawings, when each side of the "rectangular design" has *n* ends instead of 4?
* How many of these are symmetric, as with the *lusona* in figure 3.3?

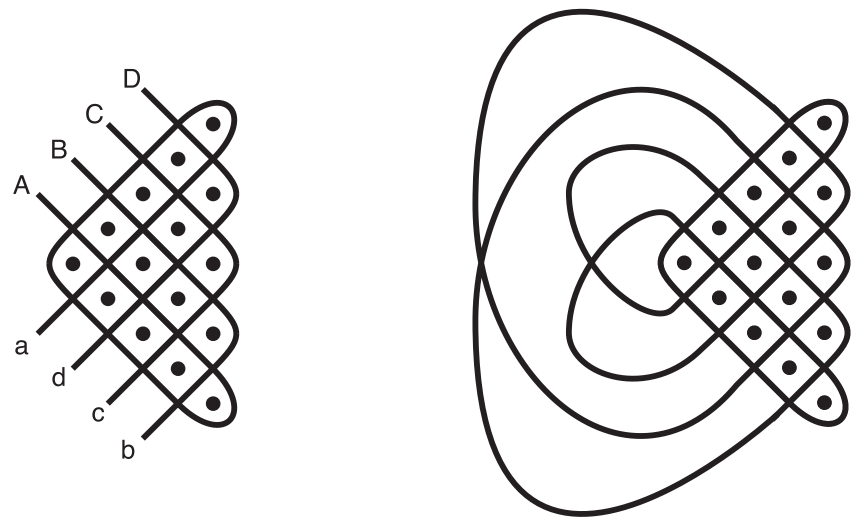


Figure 3.4

Second example

The *lusona* of figure 3.5 represents a bird in flight. It is symmetrical and monolinear. How can we now represent three birds flying in a V formation, so that the figure is both symmetric and monolinear?

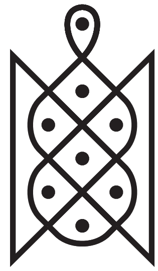


Figure 3.5

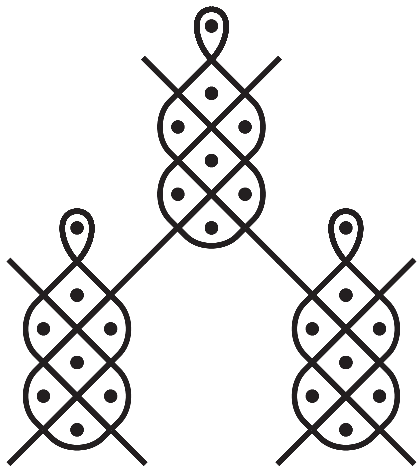


Figure 3.6

The basic pattern can be the one shown in figure 3.6. By connecting the ends, as shown in figure 3.7 one obtains, in fact, a symmetrical design, but this design is 2-linear (see figure 3.8). If we call the ends of the first line of a basic pattern **a** and **A**, those of the second line **b** and **B**, those of the third **c** and **C** and those of the fourth, **d** and **D** respectively (see figure 3.9) and linking these ends by the aAdDbBCca sequence (see the diagram in figure 3.10) we can see that we get a monolinear and symmetrical drawing (see figure 3.11). We can then ask the following questions: are there other solutions? How can we depict larger formations of Vs?

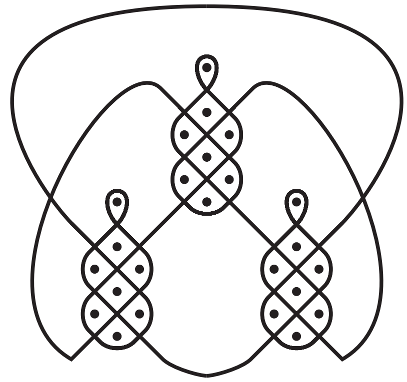


Figure 3.7

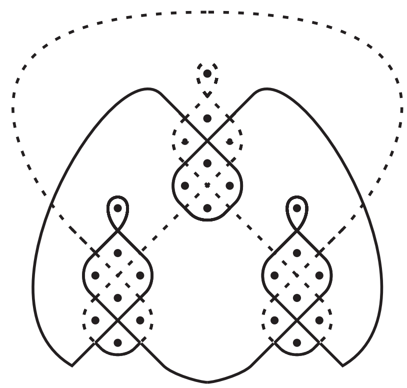


Figure 3.8

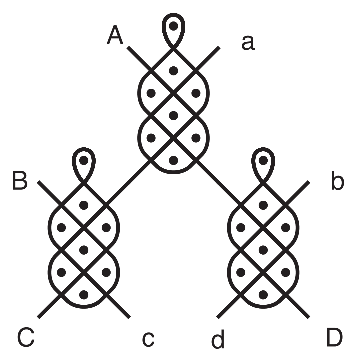


Figure 3.9

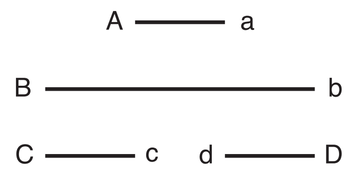


Figure 3.10

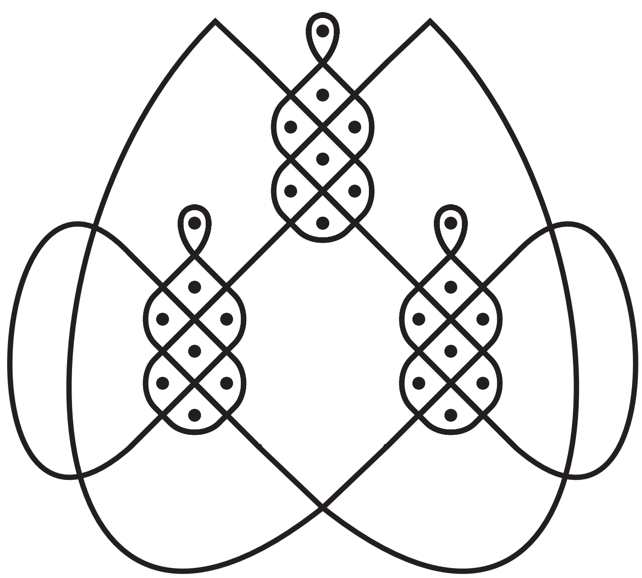


Figure 3.11

Systematic analysis of the base pattern (see figure 3.12) of a formation of 3 rows of birds allows us to conclude that there is no pattern which is both symmetrical and monolinear that represents six birds flying in a filled-V formation. Can the case analysis of 10 birds in 4 rows be more successful? The basic pattern (figure 3.13) may be completed as shown in figure 3.14, using the sequences aAHhdDEea and bBGgcCFfb. The design is symmetric and 2-linear. Following the order aAHhdDEebBGgcCFfa, we obtain a monolinear figure, which, however, is not symmetrical (see figure 3.15). Is there a solution, which is both symmetrical and monolinear?

Are there any filled-V formations of birds in 5 or more rows, which are both symmetrical and monolinear? [2]

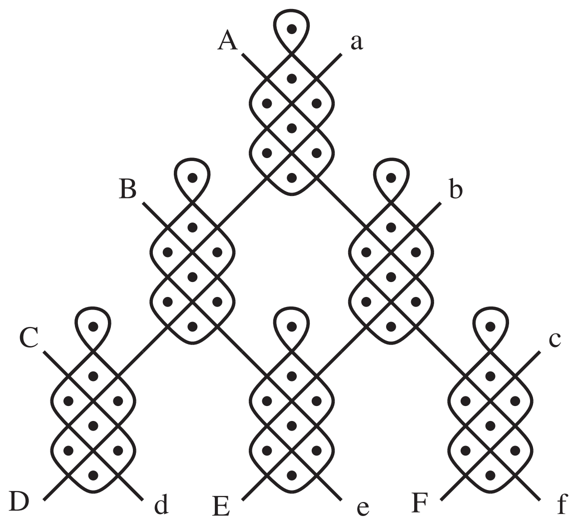


Figure 3.12

[2] The translator has provided a proof, in appendix 1, that no such formation is possible, but shows students how to create "open-V" formations which are both symmetric and monolineal.

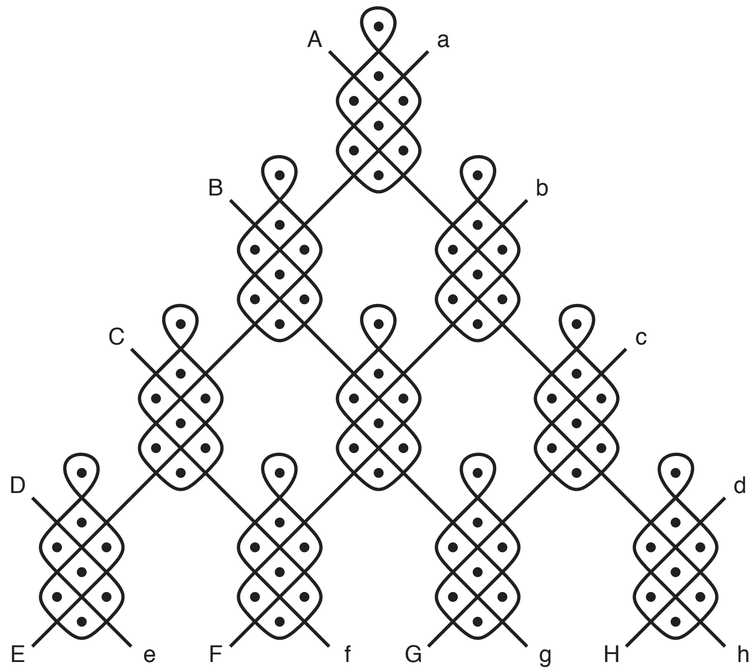
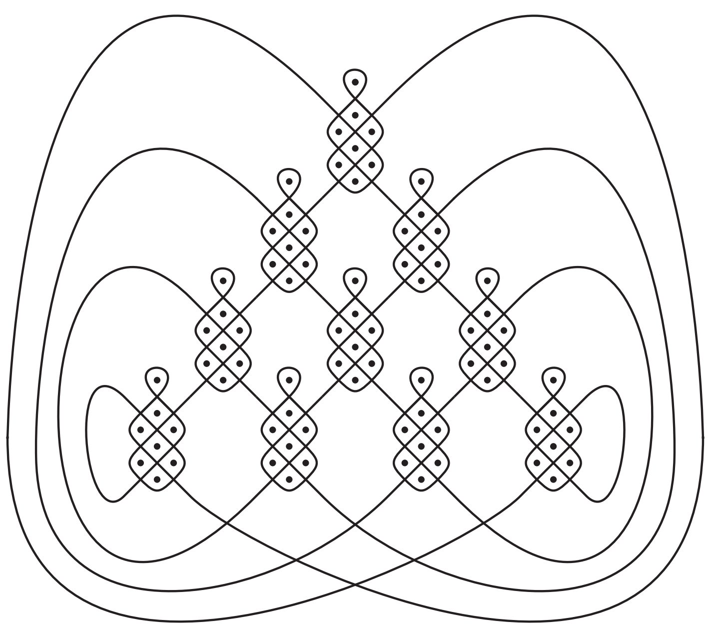
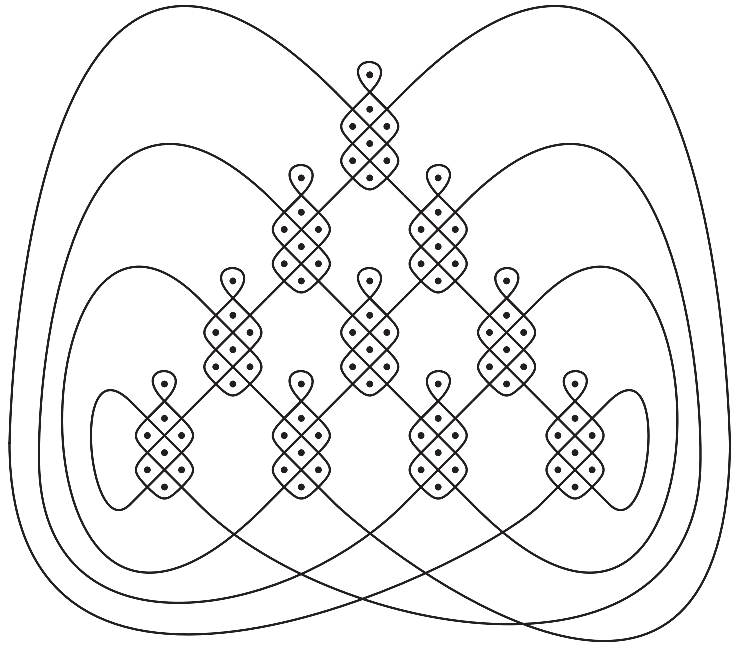


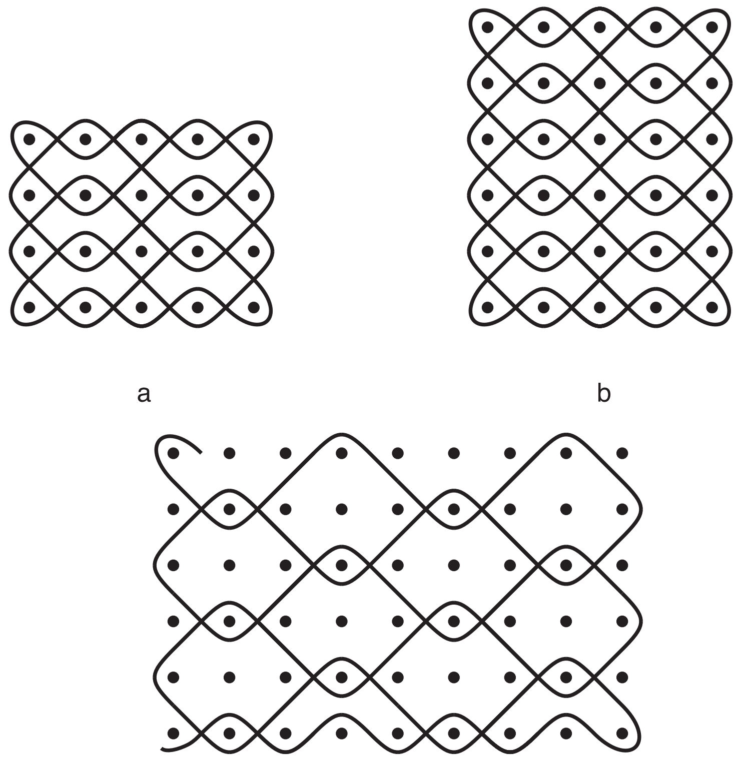
Figure 3.13



Symmetric and 2-linear design
Figure 3.14



Monolinear and almost symmetrical design
Figure 3.15



a b

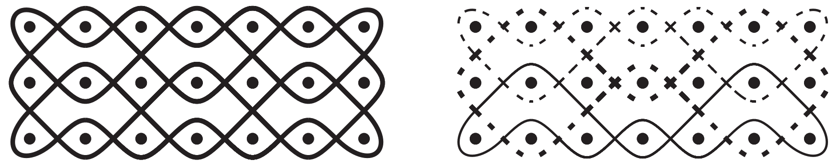
c

Figure 3.16

How many lines are needed?

* *First example*

The Cokwe ***lusona*** of figure 3.16a represents the lion's stomach. Its dimensions are 4 x 5 and it is monolinear (see Vol. 1, figure 123). Figure 3.16b has dimensions 6 x 5. The same algorithm was applied to two designs (see figure 3.16c). When using the same algorithm in the case of a reference grid having dimensions of 3 x 7, three lines are necessary to enclose all points of this grid (see figure 3.17). One might ask, how does the number of lines depend on the dimensions *m* and *n* of the grid?



a b

Figure 3.17

When the width (n) is an even number, the picture does not look like the lion's stomach (see figure 3.18). Therefore we have to consider only odd numbers for n. For the same reason, the height m must be equal to or greater than 2.

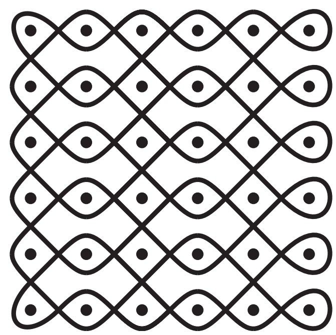


Figure 3.18

Students and future teachers can try replacing m and n with concrete numbers, draw figures, counting the lines and collect the data thus obtained on a chart like the one presented below.

	n	3	5	7	9	11	13	15	17
m									
2		2	1	2	1	2			
3		3	1	3	1				
4		4	1	4					
5		5							
6		6			1				
7									
8									
9									

The extrapolation on the basis of this experimental data may lead to the following table:

	n	3	5	7	9	11	13	15	17
m									
2		2	1	2	1	2	1	2	1
3		3	1	3	1	3	1	3	1
4		4	1	4	1	4	1	4	1
5		5	1	5	1	5	1	5	1
6		6	1	6	1	6	1	6	1
7		7	1	7	1	7	1	7	1
8		8	1	8	1	8	1	8	1
9		9	1	9	1	9	1	9	1

and the statement of a hypothesis such as:

> The number of lines in a "lion's stomach" needed to enclose all points of a grid for dimensions m x n is equal to 1 if $n = 4p+l$ and equal to m if $n \neq 4p+l$, where p represents any natural number.

Now, students and future teachers can test this hypothesis. For example, is this assumption correct in the case where $m = 4$ and $n = 13$ (see the monolinear drawing of figure 3.19)?

The next question is: how to prove this hypothesis? [3]

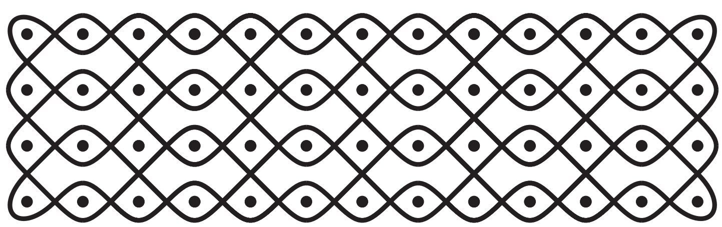


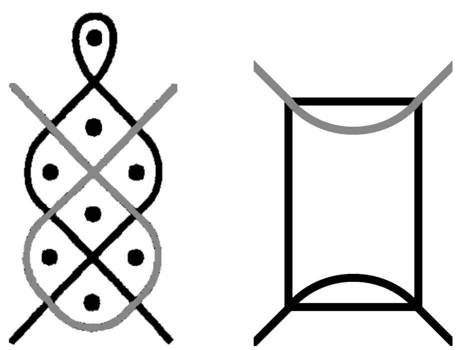
Figure 3.19

[3] See Appendix 2 for a proof of this conjecture.

APPENDIX 1
SONA OF BIRDS FLYING IN FORMATION

In this appendix, we explain why there are no monolinear, symmetric *sona* designs built from 3 or more full rows of the birds as shown in figures 3.5–3.15. However, if we look at "V" shapes of these birds, it turns out that there are many such *sona* that can be built from those layouts.

To simplify our analysis, instead of the full bird of figure 3.32(a), we use the simplification of the box of 3.32(b), which shows the entrance and exit lines for the two curves (gray and black) that make up the bird, but without the turns they take to get there. Although less artistic, such boxes make it easier to think about where the lines will be going in a larger drawing.

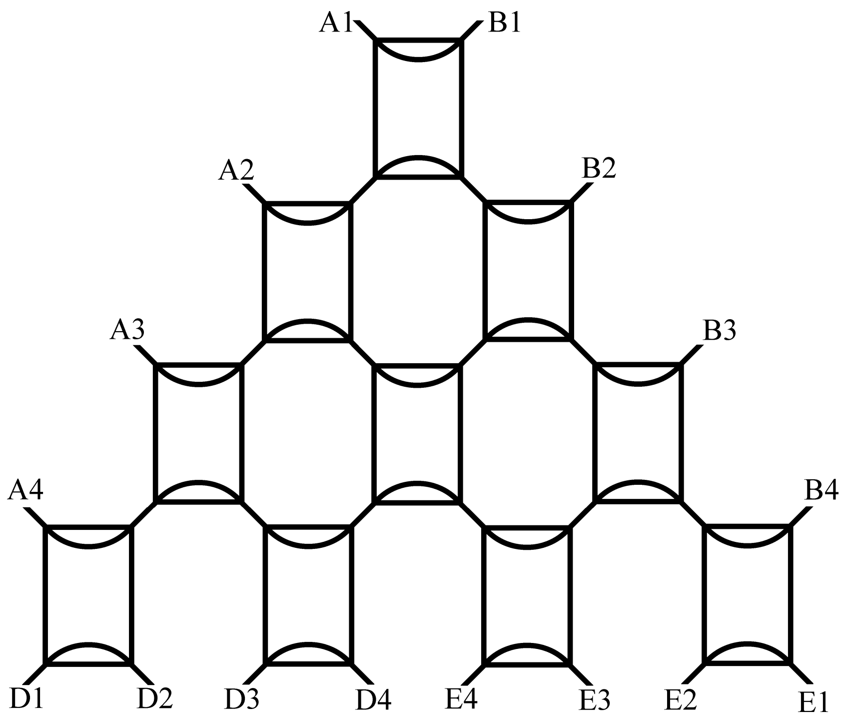


a b

Figure 3.32

To consider a formation of many birds, we arrange copies of our box figure with the connections between boxes that are mandatory, and label the remaining edge segments as shown in figure 3.33. We assume that the number "*n*" of edges of type A, B, D, and E is at least

3, corresponding to at least three rows of birds. Our goal is then to add curves connecting these various endpoints to create the desired type of *lusona*.



A flock of 4 rows of birds
Figure 3.33

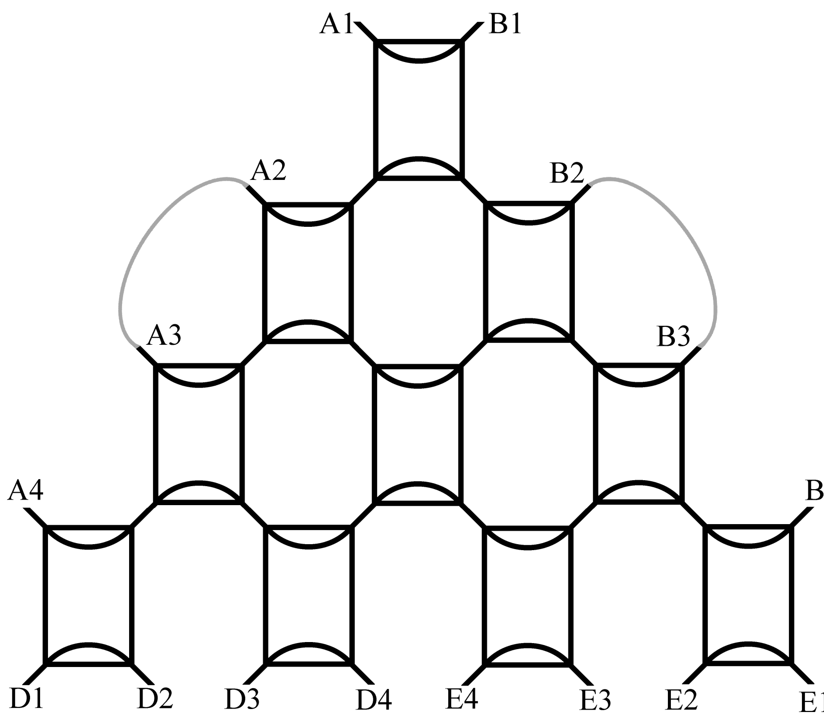
Now we imagine that there is some collection of connecting curves that would result in a monolinear, symmetric *lusona*, and we try to determine what properties those additional curves must have. First we show:

Lemma A.1: Every side edge, A_i or B_i, in a "flock of birds" *lusona* must connect to a bottom edge, D_j or E_j.

Proof: We use a proof by contradiction. Instead of the claim, imagine there is a connecting curve from A_i to A_j (See figure 3.34 for an example with i = 2 and j = 3.) Then since the final design must be symmetric, we must also have a connection from B_i to

B_j. But then we can see from figure 3.33 that A_i is also connected to B_i, and A_i to B_j. So then we would have a closed loop that travels from A_i to A_j, across to B_j, to B_i and then back to A_i. This loop of our drawing only connects two rows, *i* and *j*, hence cannot include all of the birds.

The other way we might connect two side edges is if A_i were connected to a B_j. Then, by symmetry, A_j also connects to B_i. Much like before, this creates a closed loop that travels from A_i to B_j, across to A_j, to B_i, and then back to A_i. Once again, this falls short of drawing in all of the birds.



Side edges cannot be connected
Figure 3.34

Of course since there are as many “side edges” as there are “bottom edges”, this also means:

Corollary: Every bottom edge, D_i or E_i, in a "flock of birds" *lusona* must connect to a side edge, A_j or B_j.

This is enough information to allow us to prove the conjecture about flocks of birds:

Theorem A.2: All "flocks of birds" *sona* with 3 or more rows are either asymmetric or else are *m*-linear for $m \geq 2$.

Proof: Imagine that there was a connection from, say, A_i to D_j. Then by symmetry we would also have a connection from B_i to E_j. As an example, figure 3.35 shows this situation with i =3 and j = 5. Now a curve coming into D_j flows through to D_k, where k = j ± 1, hence (by symmetry) E_j flows through to E_k for the same k. Then by the corollary, D_k and E_k are both connected to the A and B of some second row, say row *r*. Now we have a closed loop that travels from A_i to D_j to D_k to $(A \text{ or } B)_r$ to $(B \text{ or } A)_r$ to E_k to E_j to B_i to A_i. Since this loop only hits two of the "A–B" rows (rows i and r) plus two birds in the bottom row, this circuit cannot reach all of the birds in the flock.

We conclude that a full flock of birds cannot be drawn with a symmetric, monolinear *lusona* no matter how we add curves to connect the various elements. However, we can draw such *sona* with very similar configurations. Consider a "V" of such birds, as shown in figure 3.36. Now it will always be possible to draw a symmetric, monolinear *lusona* that contains all of these birds, and there will be many ways to do it! In fact, it's possible to count exactly how many mathematically different such *sona* there can be.

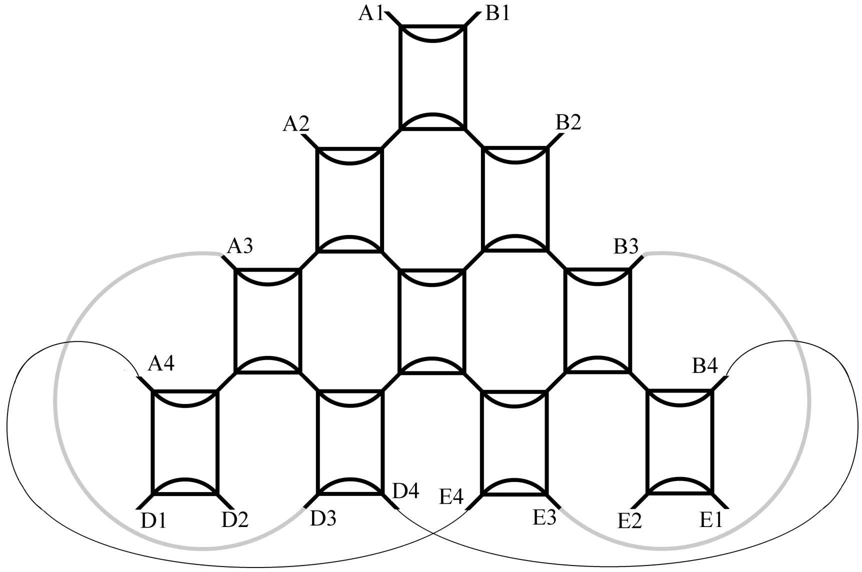
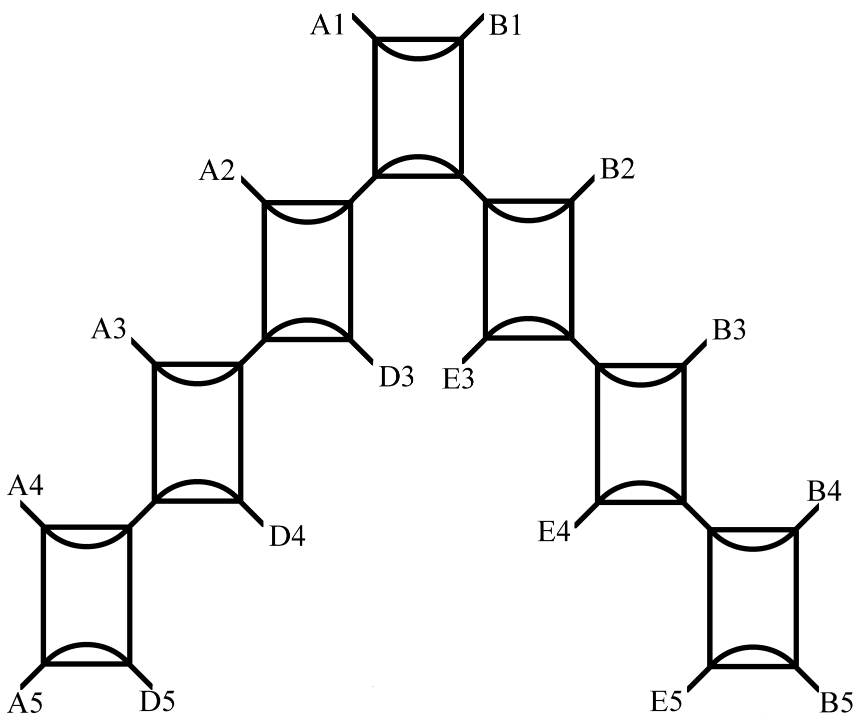
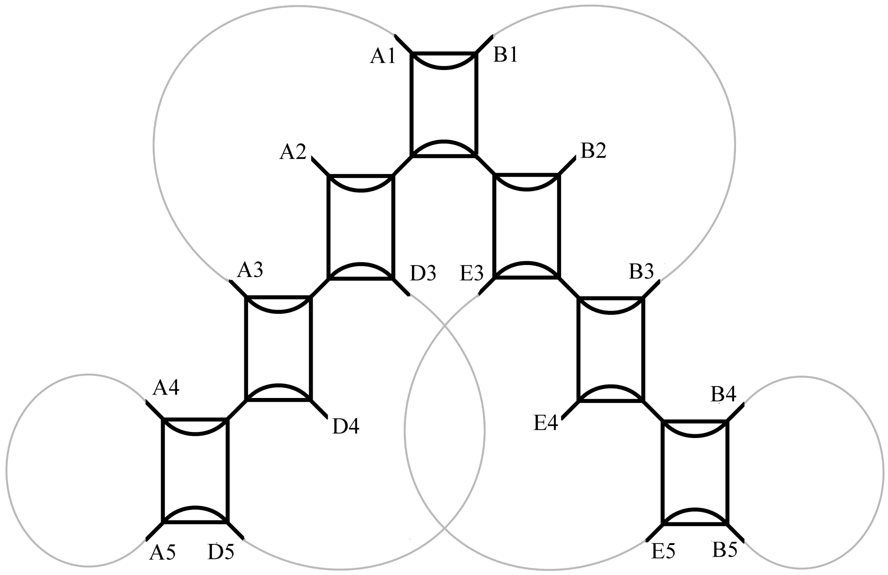


Figure 3.35

In the "V" of birds in figure 3.36, let us consider where the edges from A_1 and B_1 can connect to. If either one connects to A_2, then by symmetry the other one connects to B_2, and we get a closed loop in rows 1 and 2. In fact, if we imagine one hand drawing from A_1 while the other draws symmetrically from B_1, then when we reach that second row, we will close a loop—our two hands will reach the center of row 2 at the same time. When this happens, we will be finished drawing. To prevent this from happening too soon, we must traverse *all* of the curves on the other rows of birds before we return to row 2. So imagine drawing with one hand from A_1 to any one of the open edges on row *i* (Figure 3.37 shows an example where i = 3). By symmetry, our other hand draws from B_1 to the symmetric open edge on row *i*, our two hands traverse the two birds on row *i*, and *all* of row *i* will now be part of our design. So now we go from those 2 edges on row *i* to *any* symmetric pair of edges on row *j* (row 5 in figure 3.37), again drawing all of row *j*, etc. Every time we go from one row to the next, we have four choices for where the "A" drawing hand can go (any of the four edge connections on that row), and the choice where the "B" drawing hand will then be forced. To make sure we never reach row 2 until we have drawn all of the other *n* rows, we need to find a sequence of all the numbers from 3 to n+1 (since there are n+1 rows of open edges with n rows of birds) without repeating a number, at which point we will return to row 2 (with 2 choices for where hand "A" goes). In figure 3.37, for example, we have chosen the sequence to be 1, 3, 5, 4, 2. A valid sequence here must start with 1, end with 2, and has all of the numbers 3 … n+1 in between, which can be done in (n–2)! different ways. We have left the final choice in figure 3.37, moving from row 4 to row 2, unspecified to emphasize the two options.



A "V" of 4 rows of birds
Figure 3.36



A "V" of 4 rows of birds (5 rows of curves)
Figure 3.37

Since each time we move to a new row, other than row 2, we have 4 choices of how to connect, and we make this choice (n–1) times, we will have $4^{(n-1)}$ possible choices of edges as we travel across all of these rows, and 2 choices when we finally go to row 2 (i.e., do we go to A_2 or B_2?). The total number of ways we can select a sequence of rows to visit, combined with which of the edges on that row we go to, tells us that the number of ways to draw such a *lusona* will be:

$$2 * 4^{(n-1)} * (n-1)!$$

The first few values of this number are:

Rows of birds:	2	3	4	5
Number of *sona*:	8	64	768	12,288

So there are many ways to draw a V of birds with symmetric, monolinear *sona*! The eight *sona* with a 2-row V design of birds are easy to describe: Start at A_1, then go to any of the 4 connections in the 3^{rd} row of curves, pass through that bird, then go to either of the 2 connections in the 2^{nd} row of curves, for a total of 4*2 possibilities. Once you decide where to draw the curve from A_1, you also know what happens with the curve from B_1, by symmetry. Of course to find a way to draw these curves so that they are artistically pleasing is still a challenge! We invite the student to look at the examples in figures 3.7 to 3.15 and try to find a particularly pleasing way to draw one of these 8 options. Re-drawing the two *sona* implied by figure 3.37 using something other than circular arcs (as shown here) can provide artistic options itself. Then, for a bigger challenge, try to find an artistically pleasing *lusona* from one of the larger “V” formations, e.g. with 3, 4, or 5 rows of birds in addition to those of figure 3.37.

APPENDIX 2

LION'S STOMACH *SONA*

In this appendix, we study the Lion's Stomach sona in more detail, leading to a proof of the author's conjecture about the number of lines needed for such a sona. But first we must take a side trip to investigate some properties of "sona design strips". By this, we imagine taking a rectangular $n \times m$ lusona design and cutting it into pieces with parallel lines midway between columns (or rows) of dots. The lusona S of figure 1, for example, has been cut into the pieces A, B, C, and D by the three lines shown, and we write this equivalence as S = ABCD.

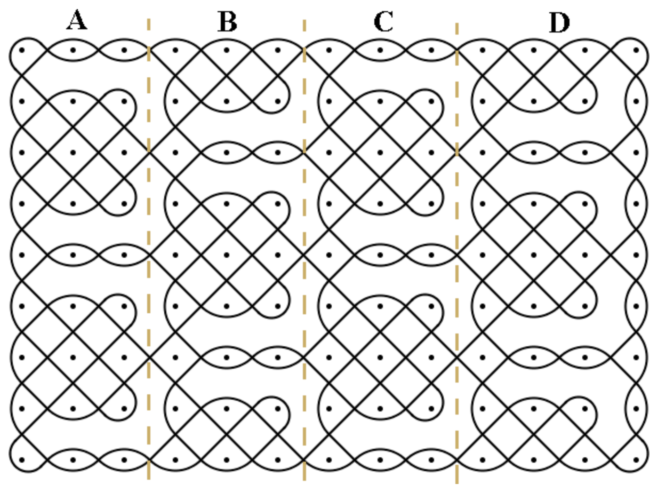


Figure 1: A lusona cut into 4 "strips", A, B, C, and D

In some cases, it is possible to take a pair of strips of the same height and combine them to create either a larger strip or a proper lusona. To combine strips X and Y (of the same height) to form the larger strip XY, the right side of X and the left side of Y must have sets of walls in identical locations. In the example of figure 1, all 3 cutting lines hit walls in exactly equivalent places, hence we can combine these segments in several ways to create various sona including AD, ABD, ACD, ABBD, etc. Figure 2 shows the lusona that is equivalent to ABBD, which we also write as AB^2D.

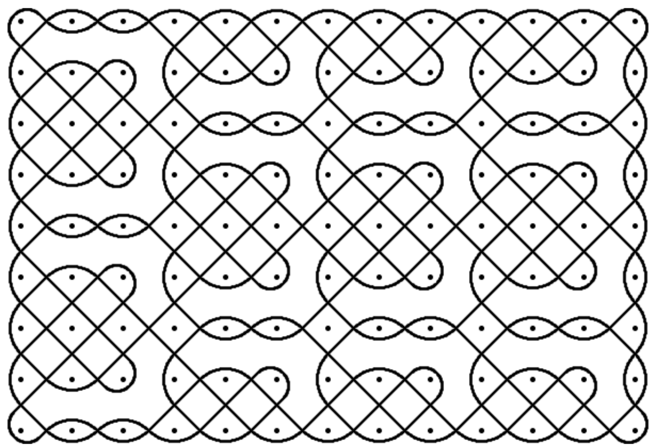


Figure 2: The same strips combined as ABBD

Although the original ABCD and this ABBD are both legitimate sona designs, ABCD is a one-line sona, while AB^2D is a 3-line sona. Under certain circumstances, though, it is possible to guarantee that certain combinations of these strips will produce one-line sona. The primary result here is:

The Pumping Lemma: Let S be a rectangular k-line lusona cut into three strips by the two parallel lines L_1 and L_2, so that S = ABC, and where:

1. The intersections of L_1 and L_2 with S meet walls in identical locations;
2. There is no line of S that is completely contained inside B; and
3. Every line of S that enters B on the left exits it on the right at the same height, and traveling in the same direction, i.e. either both are directed 45° up (to the right) or 45° down (to the right).

Then all lusona $S_i = AB^iC$ are also k-line sona, for i = 0, 1, 2, ….

We call this the "pumping lemma" (after a similarly named theorem in computer science) because additional copies of "B" can be "pumped in" to the inside of the lusona, or the single copy in the original can be "pumped out".

Proof: By condition #1, the number of lines coming into B on the left must be exactly the same as the number leaving B on the right. With condition #3, this means that every line in B that meets the right edge of B must have come from an edge that entered B on the left. From this we can conclude that there can be no line of S_i that is completely contained inside the sequence of strips B^i. If there were, then that line would have a left most strip B_j through which it traveled, and hence by condition #2 it would also travel through strip B_{j+1}, which would violate our second sentence.

Now each line of S that enters B on the left at height h also exits it at height h (see figure #3 for some possible examples). Then with several copies of B in succession, that line would enter the left most edge of B^i at height h and leave the full B^i on the right at the same height. Thus regardless of the number of copies of B (including none), lines in A and C will be connected in exactly the same ways, in exactly the same order. Thus these lines, and their connecting segments through B^i, will create the same number of total lines as in S. By the previous paragraph, there can be no other lines of S_i not included in this count, hence S_i is also a k-line lusona. ■

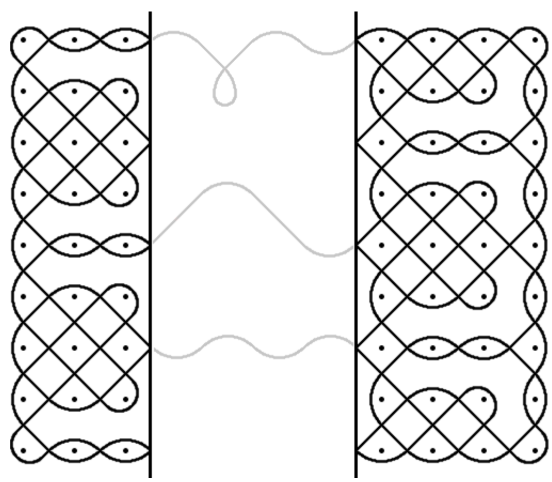


Figure 3: Sona lines passing through strip B.

We can use this lemma to analyze many different classes of sona designs, including the author's conjecture on the "Lion's Stomach" class of sona. These designs exist for any rectangle with an odd number of columns and at least two rows of dots. Gerdes conjectures that if a "Lion's Stomach" design is built on a sona grid of width n = 2k + 1 and height m = j, then the number of lines required for the lusona will be 1 when k is even, and will be m when k is odd. We will show why this is true.

Figure 4 shows the 6 x 13 Lion's Stomach lusona in two views: The top shows the lusona without the walls, as it would be drawn in the sand by the Cokwe/Chokwe. The bottom shows the lusona with the walls that imply the drawing of the design, but here we also show two copies of a central strip that meets the requirements of the Pumping Lemma. For ease of verification, we have shown only four of the lines that go across these strips; all of the undrawn lines can be seen (by the pattern of the walls) to be equivalent to one of these. The Pumping Lemma tells us that we can remove all of these strips from Lion's Stomach sona without changing the number of lines in these sona.

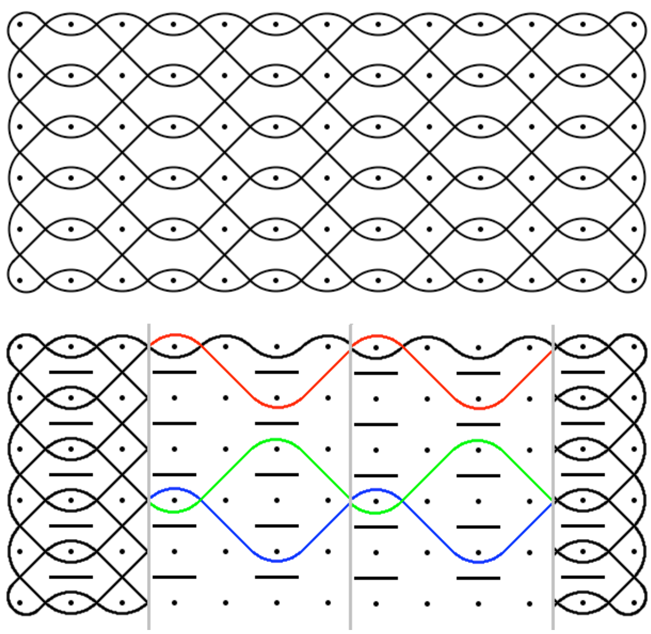


Figure 4: The"Lion's Stomach" sona meets the conditions of the Pumping Lemma with a strip of width 4.

This tells us that:

Corollary (to the Pumping Lemma)**:** If S is a Lion's Stomach lusona of width $n = 4k + 1$ and height $m = j$, then it requires the same number of lines to draw as a Lion's Stomach lusona of width 5 and height j. If S is a Lion's Stomach lusona of width $n = 4k + 3$ and height $m = j$, then it requires the same number of lines to draw as a Lion's Stomach lusona of width 3 and height j.

Thus we need only understand these designs for width 3 and 5, and we will understand all such designs. For convenience, we show the 3 x 8 and 5 x 8 Lion's Stomach designs in figure 5.

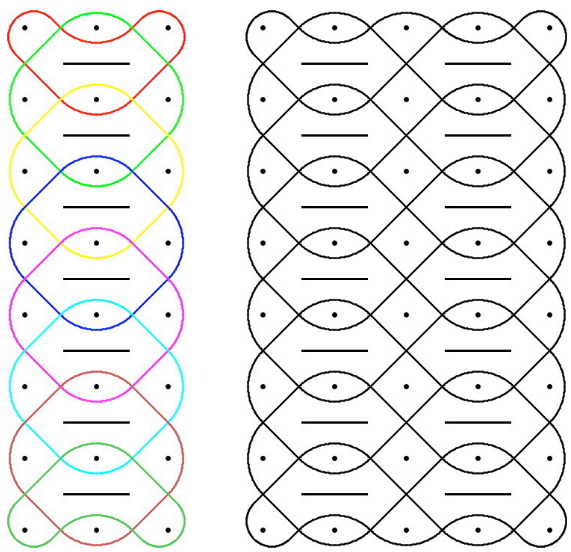


Figure 5: Lion's Stomach sona of sizes 3 x 8 and 5 x 8. With the 3 x 8 lusona, multiple lines are shown in colors. The 5 x 8 is a one-line lusona.

It is easy to see the pattern in the 3 x j Lion's Stomach that results in it requiring j lines to be drawn, demonstrating this part of Gerdes' conjecture. The 5 x 8 lusona is monolineal. Figure 6a shows a continuous segment of this line colored in green, which is the only portion of the curve which traverses the bottom row of dots of the lusona. If we replace this green segment with the one in figure 6b, which starts and ends in the same place, we have drawn the 5 x 7 Lion's Stomach lusona using the same number of lines, i.e. one line.

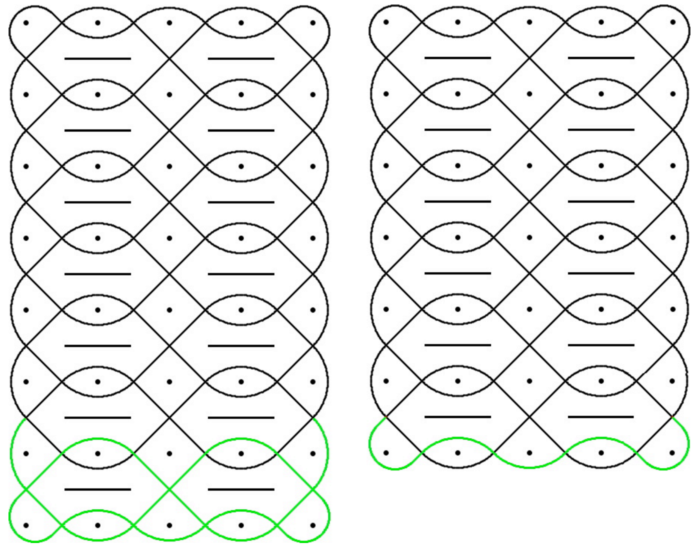


Figure 6: Lion's Stomach sona of sizes 5 x 8, showing the reduction to the 5 x 7 Lion's Stomach.

Thus we have proved Gerdes' conjecture:

Theorem: "Lion's Stomach" sona of width 4k+1, regardless of height, are always monolineal sona. A "Lion's Stomach" lusona of width 4k + 3 and height *m* = j is a j-line lusona.